Problem 8.4.7

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02407 Stochastic Processes

We model the spot price process of a stock with a geometric Brownian motion $\{Z_t\}_{t\geq 0}$ with drift parameter α and volatility (variance) parameter σ^2, and we know that the current market price is $Z_0 = z > 0$ (we let the current time be $t = 0$). We then consider a call option on said stock with strike price $K > 0$ (in the book, they call it a) and maturity (expiration date) τ time units from $t = 0$. We can then calculate the probability that the option is in the money at maturity as

$$\mathbb{P}(Z_\tau > K | Z_0 = z) = \mathbb{P}(\ln(Z_\tau) > \ln(K) | \ln(Z_0) = \ln(z)).$$

From p. 424, sec. 8.4.2, we know that the process $\{\ln(Z_t)\}_{t\geq 0}$ is Brownian motion with drift $\mu = \alpha - \frac{1}{2}\sigma^2$ and variance parameter σ^2. Therefore, we may apply the formula on p. 419, which yields

$$\begin{aligned}
\mathbb{P}(\ln(Z_\tau) > \ln(K) | \ln(Z_0) = \ln(z)) &= 1 - \mathbb{P}(\ln(Z_\tau) \leq \ln(K) | \ln(Z_0) = \ln(z)) \\
&= 1 - \Phi\left(\frac{\ln(K) - \ln(z) - \mu\tau}{\sigma\sqrt{\tau}}\right) \\
&= 1 - \Phi\left(\frac{\ln(K) - \ln(z) - \left(\alpha - \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}\right) \\
&= 1 - \Phi\left(\frac{\ln(K/z) - \left(\alpha - \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}\right).
\end{aligned}$$